Math 250
Fall 2013
Final exam
12/16/13

Name (Print): ____________________

This exam contains 12 pages (including this cover page) and 16 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	10	
3	10	
4	15	
5	10	
6	15	
7	15	
8	15	
9	15	
10	20	
11	15	
12	20	
13	10	
14	15	
15	3	
16	3	
Total:	206	

1. (a) (10 points) Find the RREF of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ -1 & -2 & 1 & 2 \\ 2 & 4 & -3 & 2 \\ -3 & -6 & 2 & 0 \end{bmatrix}$$

Ans:

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) (5 points) What are its rank and nullity? Rank = 3, Nullity = 1.

2. (10 points) Let R be the RREF of the matrix A given below. Find the invertible matrix P so that $PA = R$.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}$$

Ans: Consider the matrix

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right].$$

Row reduce it we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right].$$

So

$$P = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}.$$

3. (10 points) A Rutgers student has to take certain courses in Art, Science and Math to fulfill the requirements for his degree. There are three categories that are required for the degree: Critical thinking, Quantitative reasoning and Appreciation of life. Taking a course in Art, Science or Math counts differently in terms of units toward these different categories. The break down is as followed:

	Art	Science	Math
Critical Thinking	1 unit	2 units	2 units
Quantitative reasoning	1 unit	3 units	3 units
Appreciation of life	4 units	1 unit	1 unit

The student needs to take exactly 6 units of Critical thinking, 8 units of Quantitative reasoning and 10 units of Appreciation of life. He also likes Math very much. What is the maximum amount of Math courses can he take to provide exactly the required units for each category? Ans:

Consider the augmented matrix

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 6 \\ 1 & 3 & 3 & 8 \\ 4 & 1 & 1 & 10 \end{array}\right].$$

Row reduce it we get

$$R = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore he can take at most 2 math courses.

4. Let

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}; B^{-1} = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 4 \\ 3 & -2 & 1 \end{bmatrix}; C^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Compute the following

(a) (5 points) $(A^T B^T)^{-1}$

Ans: $(A^T B^T)^{-1} = (B^{-1})^T (A^{-1})^T = \begin{bmatrix} 5 & 3 & 2 \\ -3 & -2 & -1 \\ 4 & 5 & 7 \end{bmatrix}$

(b) (5 points) $(CA^T)^{-1}$ Ans: $(CA^T)^{-1} = (A^{-1})^T C^{-1} = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 1 & 0 \\ 3 & 2 & 4 \end{bmatrix}$

(c) (5 points) $(BAC^T)^{-1}$ Ans: $(BAC^T)^{-1} = (C{-}1)^T A^{-1} B^{-1} = \begin{bmatrix} 13 & -8 & 21 \\ 12 & -7 & 15 \\ 16 & -10 & 10 \end{bmatrix}$

5. (10 points) Determine, if possible, a value of r for which the given set is **linearly dependent**.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} r \\ 0 \\ r \end{bmatrix}, \right\}$$

Ans: $r = 0$ since that would make the last column the zero vector.

6. (a) (5 points) Find all values of c so that the following matrix **i**s invertible.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 3 & 4 & 7 \end{bmatrix}$$

Ans: $DetA = -10 + 2c$ so $c \neq 5$.

(b) (10 points) For the values of c in part a, find the solution to the following system of equations (in terms of c):

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ c \\ 1 \end{bmatrix}$$

Apply Cramer's rule: $x_1 = -2, x_2 = 0, x_3 = 1$.

7. Let

$$A = \begin{bmatrix} 1 & -2 & 0 & 2 \\ -1 & 2 & 1 & -3 \\ 2 & -4 & 3 & 1 \end{bmatrix}$$

(a) (5 points) Find a basis for Col A.
Ans: The REF of A is

$$R = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So a basis is $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

(b) (5 points) Find a basis for Null A. A basis is $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

(c) (5 points) Find a basis for Row A. A basis is $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$

8. Let

$$A = \begin{bmatrix} -2 & 6 & 3 \\ -2 & -8 & -2 \\ 4 & 6 & -1 \end{bmatrix}$$

(a) (5 points) Find the eigenvalues of A (Hint: You may want to consider row-reducing the matrix $A - \lambda I$ before looking for the eigenvalues).

Ans: The eigenvalues are: -2,-4,-5.

(b) (5 points) For each of the eigenvalue found in part a, find a basis of the corresponding eigenspace.

For $\lambda = -2$, a basis is $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

For $\lambda = -4$, a basis is $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

For $\lambda = -5$, a basis is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(c) (5 points) Find an invertible matrix P and a diagonal marix D such that $A = PDP^{-1}$.

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -5 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

9. (15 points) Let

$$A = \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix}$$

Find A^k for an arbitrary integer k.

Ans: A has the following diagonalization

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$$

So

$$A^k = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} (-2)^k & 0 \\ 0 & (-3)^k \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2(-3)^k - (-2)^k & (-2)^k - (-3)^k \\ 2(-3)^k - 2(-2)^k & 2(-2)^k - (-3)^k \end{bmatrix}$$

10. (20 points) Determine all values of c so that the following matrix is **n**ot diagonalizable.

$$\begin{bmatrix} -3 & 0 & -2 \\ -6 & c & -2 \\ 1 & 0 & 0 \end{bmatrix}$$

Ans: The eigenvalues of the matrix are: $c, -1, -2$. If $c = -1$ then

$$A - cI = \begin{bmatrix} -2 & 0 & -2 \\ -6 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix}.$$

Reducing we get

$$R = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So this matrix has rank 2 and for the eigenvalue -1 there is only 1 eigenvector, so it's not diagonalizable.

If $c = -2$ then

$$A - cI = \begin{bmatrix} -1 & 0 & -2 \\ -6 & 0 & -2 \\ 1 & 0 & 2 \end{bmatrix}.$$

Reducing we get

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So this matrix also has rank 2 and for the eigenvalue -2 there is only 1 eigenvector, so it's not diagonalizable.

Conclusion: For $c = -1, -2$ the matrix is not diagonalizable. Note: You need to show all the above arguments to get full credit for this question. It is not enough to just compute the eigenvalues and conclude $c = -1, -2$.

11. Let S be the following set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \right\}$$

(a) (5 points) Apply the Gram-Schmidt procedure to replace the set S by an orthogonal set of non-zero vectors with the same span. Ans:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \right\}$$

(b) (5 points) Use the result in part a to find an orthonormal set of non-zero vectors with the same span as S.

$$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{15}} \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \right\}$$

(c) (5 points) Let $u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find the orthogonal projection of u onto the subspace spanned by S.

The projection is:

$$w = \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

12. In this question, matrices A, Q and R and a vector b are given. Verify if Q and R are indeed the QR factorization of A and if that is true, **use the QR factorization** to solve for $Ax = b$.

(a) (10 points)

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix}, Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{15}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{15}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{3}{\sqrt{15}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{15}} \end{bmatrix}, R = \begin{bmatrix} \sqrt{3} & \sqrt{3} & 2\sqrt{3} \\ 0 & \sqrt{3} & \frac{-2}{\sqrt{3}} \\ 0 & 0 & \frac{5}{\sqrt{15}} \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ -7 \\ -1 \\ 11 \end{bmatrix}$$

Ans: Not the right QR factorization.

(b) (10 points)

$$A = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}, Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & \frac{3}{\sqrt{12}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{12}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{12}} \end{bmatrix}, R = \begin{bmatrix} \sqrt{6} & \sqrt{6} & 2\sqrt{6} \\ 0 & \sqrt{6} & \frac{8}{\sqrt{6}} \\ 0 & 0 & \frac{4}{\sqrt{12}} \end{bmatrix}$$

$$b = \begin{bmatrix} 8 \\ 0 \\ 1 \\ 11 \end{bmatrix}$$

Ans: Not the right QR factorization.

13. Let

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 + x_4 \\ 2x_1 - 5x_2 + x_3 + 3x_4 \\ x_1 - 3x_2 + 3x_4 \end{bmatrix}$$

(a) (5 points) Is there a non-zero vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \mathbf{0}$? Ans:

The matrix corresponding to T is

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 \\ 2 & -5 & 1 & 3 \\ 1 & -3 & 0 & 3 \end{bmatrix}$$

Its REF is

$$R = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So the matrix is NOT one to one (there is free variable). So there is a non-zero vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \mathbf{0}$.

(b) (5 points) Is there a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{2} \\ e^2 \\ \pi \end{bmatrix}$?

The matrix has full row rank, so it is onto. So there is a solution to $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{2} \\ e^2 \\ \pi \end{bmatrix}$.

14. (15 points) A set of vectors S is given. Find a basis for the subspace $S^{\perp}$.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -5 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -7 \\ 0 \end{bmatrix} \right\}$$

Ans: $S^{\perp}$ is Null A where

$$A = \begin{bmatrix} 1 & -1 & -5 & -1 \\ 2 & -1 & -7 & 0 \end{bmatrix}$$

Reducing we have

$$R = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

So a basis is

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

15. (3 points) (Extra credit) Recall that a matrix A is anti-symmetric if $A^T = -A$ and it is symmetric if $A^T = A$. Let $A_{5\times 5}$ by an anti-symmetric matrix. Is A onto?

Ans: Since $A = -A^T$, we have $Det(A) = Det(-A^T) = -Det(A)$. This implies $Det(A) = 0$ and so A is not onto.

16. (3 points) (Extra credit) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find a symmetric matrix B and an anti-symmetric matrix C so that $A = B + C$. (See the above extra credit question for definition of symmetric and anti-symmetric matrices).

Ans:

$$B = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}, C = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$